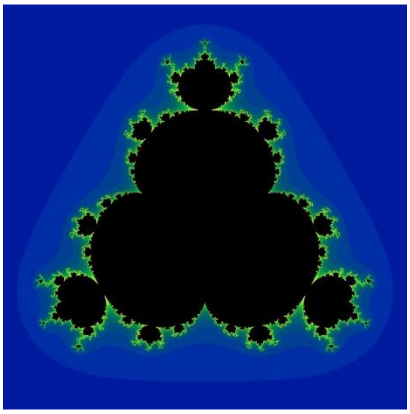
Multidimensional maps

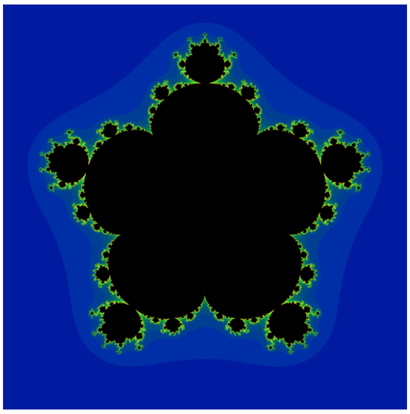
The Mandelbrot set is based on $z \to z^2 + c$. With $z_k = x_k + iy_k$ and $c = c_x + ic_y$ this can be expressed as:

$$\begin{cases} x_{k+1} = x_k^2 - y_k^2 + c_x \\ y_{k+1} = 2x_k y_k + c_y \end{cases}$$

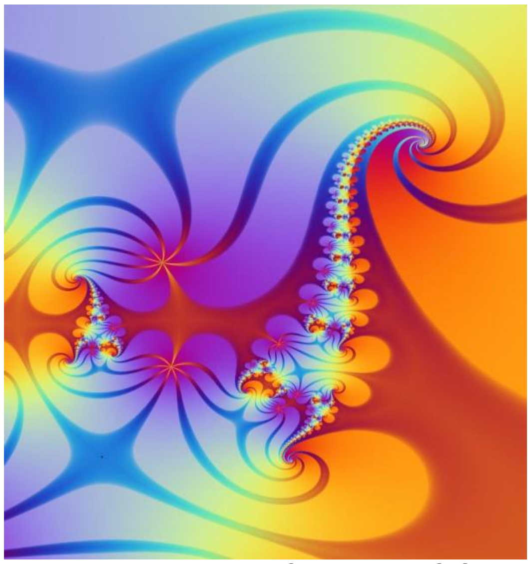
Some other 2-dimensional 'Mandelbrot sets', based on other maps are:



$z \to z^4 + c$



$z \to z^6 + c$

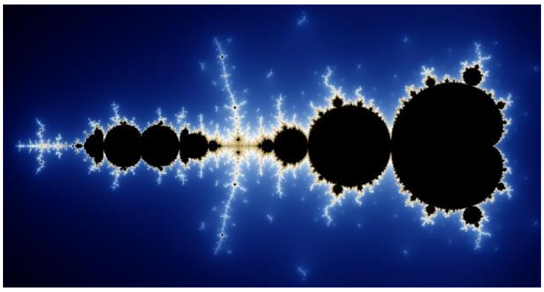


Julia set for $z \to (1 - z^3/6)/(z - z^2)^2 + c$

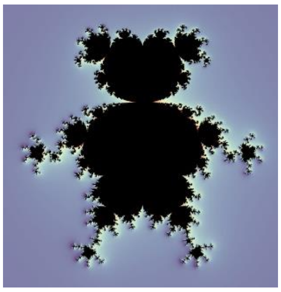
Multibrot.gif

$z \to z^d + c$ with $d: 0 \to 8$

① ② ③ $\leftrightarrow \begin{cases} x_{k+1} = x_k^2 - y_k^2 + x_k y_k + c_x \\ y_{k+1} = x_k y_k + c_y \end{cases}$

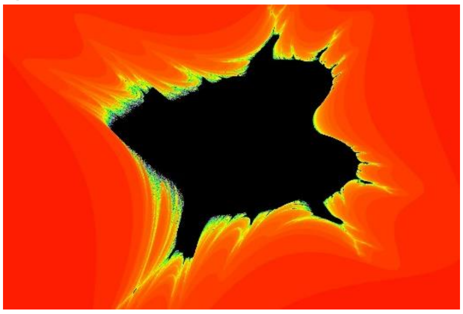


$z \to z^2 + 0.19z^3 + c$

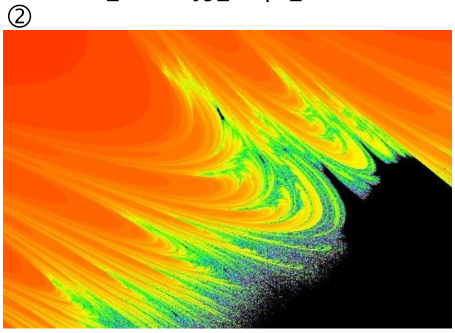


$z \to -icz^5 + 1$

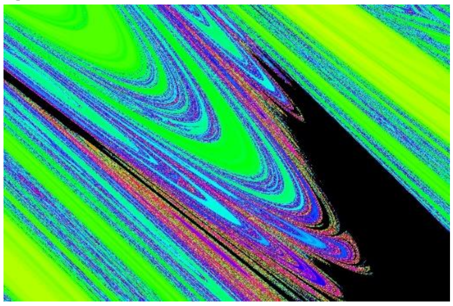
①



②



③



Stability analysis

The general form of a multidiomnensional system in n real dimensions:

$$\begin{cases} x_1(k+1) = f_1\big(x_1(k), x_2(k), \dots, x_n(k)\big) \\ x_2(k+1) = f_2\big(x_1(k), x_2(k), \dots, x_n(k)\big) \\ \vdots \\ x_n(k+1) = f_n\big(x_1(k), x_2(k), \dots, x_n(k)\big) \end{cases} \quad \text{or} \quad \boldsymbol{r}_{k+1} = \boldsymbol{f}(\boldsymbol{r}_k)$$

For a given initial value $\boldsymbol{r}_0$ the dynamics of the iteration can result in a limiting behavior that is

- A fixed point $\quad \boldsymbol{r}^* = \boldsymbol{f}(\boldsymbol{r}^*)$
- Periodic $\quad \boldsymbol{r} = \boldsymbol{f}^N(\boldsymbol{r})$
- Quasiperiodic $\quad$ Two or more incommensurable periods, Ex. Periodic in x_1 and x_2 but not in (x_1, x_2)
- Chaotic $\quad$ Orbits with exponential separation

Let two trajectories start close together $\boldsymbol{r}_0$ and $\tilde{\boldsymbol{r}}_0 = \boldsymbol{r}_0 + \delta\boldsymbol{r}_0$ with a small difference $|\delta\boldsymbol{r}_0| \ll 1$.
The one-dimensional case $\delta x_{k+1} \approx f'(x_k) \cdot \delta x_k$ will be generalized to:
$\delta\boldsymbol{r}_{k+1} \approx \boldsymbol{Df}(\boldsymbol{r}_k) \cdot \delta\boldsymbol{r}_k$ where $\boldsymbol{Df}(\boldsymbol{r}_k)$ is the Jacobian matrix $\boldsymbol{Df}$ evaluated at $\boldsymbol{r}_k$.

$$\boldsymbol{Df} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \dots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \dots & \partial f_2/\partial x_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \dots & \partial f_n/\partial x_n \end{pmatrix}$$

In two dimensions after a change of variables that diagonalizes $\boldsymbol{Df}$ with eigenvectors $\boldsymbol{v}_k$ and eigenvalues $\tilde{h}_k$:

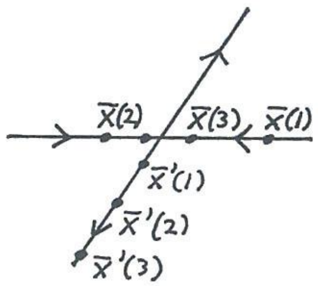
$$\begin{pmatrix} \delta x'_1(k+1) \\ \delta x'_2(k+1) \end{pmatrix} = \begin{pmatrix} \tilde{h}_1 & 0 \\ 0 & \tilde{h}_2 \end{pmatrix} \begin{pmatrix} \delta x'_1(k) \\ \delta x'_2(k) \end{pmatrix} \quad \rightarrow \quad \text{For a general pertubation of } \varepsilon_k \text{ along } \boldsymbol{v}_k \quad \delta\boldsymbol{r}_k = \sum_{i=1}^{n} \varepsilon_i \left(\tilde{h}_i\right)^k \boldsymbol{v}_i \quad (*)$$

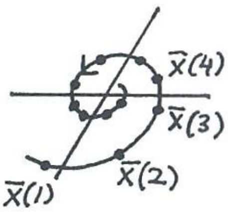
A coplication here is that the eigenvectors and eigenvalues will vary along the trajectory for non-linear iterations that have derivatives that are not constant.

$(*)$ can be used to determine the stability of a fixed point. With several directions there can be different stability properties along different directions. Stability is decided by the largest eigenvalue, labelled $\tilde{h}_1$.

Type of fixed point	Condition
Stable	$\lvert\tilde{h}_1\rvert < 1$
Unstable	$\lvert\tilde{h}_1\rvert > 1$
Undecided	$\lvert\tilde{h}_1\rvert = 1$

For complex conjugate pairs of eigenvalues, as can happen with a real Jacobian, the table still hold but the conjugate pairs $|\tilde{h}|e^{\pm i\varphi}$ will result in ingoing or outgoing spirals.



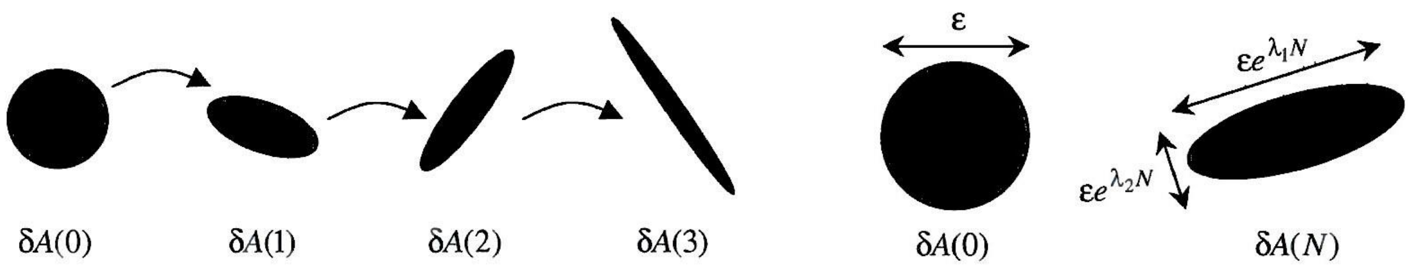


The left example has one stable and one unstable direction,

The right example is stable with complex conjugate eigenvalues with absolute value less than one resulting in an ingoing spiral towards a fixpoint.

Lyapunov exponents

Derivation of Lyapunov exponents is a bit tricky since the eigendirections change with each iteration in the $(x_1, x_2, \dots x_n)$-space. Assume exponential expansion or contraction of $\delta\boldsymbol{r}_0$ over N iterations $|\delta\boldsymbol{r}_N| \sim |\delta\boldsymbol{r}_0| e^{\lambda N}$.



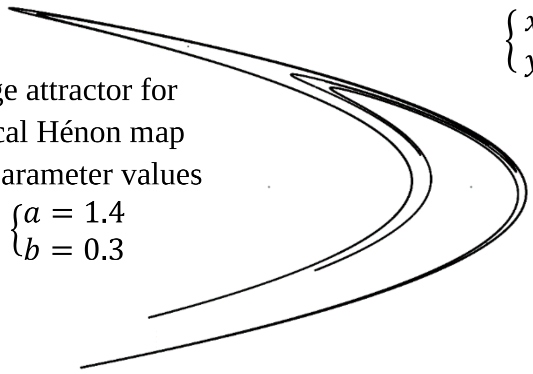
An infinetisimal circle is transormed into an ellipse, a sphere becomes an ellipsoid etc. The stretchings and compressions define a largest Lyapunov exponent λ_1.

If $\lambda_1 > 1$ there will be chaosin the long run where separation of points in $(x_1, x_2, \dots, x_n)$-space will be determined by exponential expansion in this direction.

Hénon map

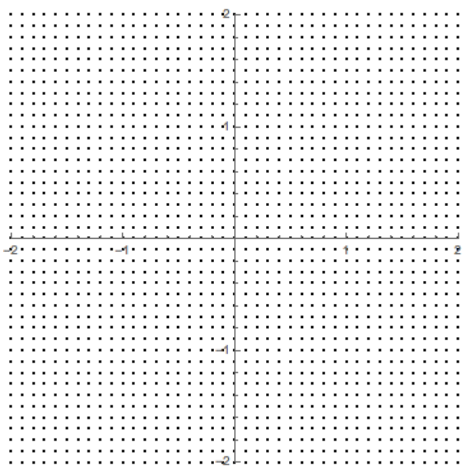
Books and studies of non-linearity and chaos often use the **Hénon map** as an example of a system with chaotic and fractal properties. It's a discrete dynamical system $\boldsymbol{r}_{n+1} = f(\boldsymbol{r}_n)$ with:

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n \\ y_{n+1} = b x_n \end{cases}$$



Strange attractor for classical Hénon map with parameter values $\begin{cases} a = 1.4 \\ b = 0.3 \end{cases}$

The limiting orbit can be periodic, intermittent or chaotic. A dynamical system is said to be intermittent when it alternates phases of periodic and chaotic dynamics. The classical Henon map with $a = 1.4$ and $b = 0.3$ will depending on the initial value $\boldsymbol{r}_0$ either diverge to infinity or approach a set called the Hénon strange attractor, a fractal and Cantor-like set with Hausdorff dimension 1.26. The attractor contains unstable periodic orbits.

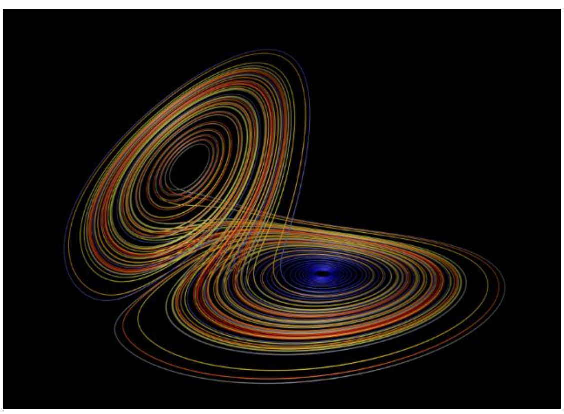


The map can be decomposed into 3 steps:
a bend → a contraction in x → a reflection in $x = y$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ 1 - ax^2 + y \end{pmatrix} \to \begin{pmatrix} bx \\ 1 - ax^2 + y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$$

The Cantor-like structure is a consequence of the repeated stretching and folding operations that are an invariant feature of the limiting strange attractor.

The Henon map is not very old, it is from 1976, a result of studies that tried to simplify the Lorentz attractor with **Poincaré maps**.
The Lorentz attractor was discovered in the 60´s in connection with ordinary differential equation used to study weather phenomena.
The attractor gave birth to the term 'Butterfly effect', due to its appearance and the saying that a butterfly can cause a storm when flapping its wings based on sensitive dependence on initial conditions.



Cat map

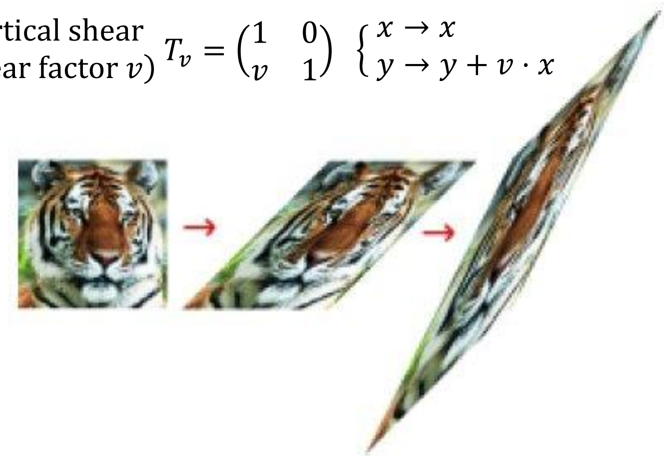
Another useful 2-dimensional map is Arnold's cat map named after Vladimir Arnold, a Russian mathematician who studied it in the 1960's. The map is often illustrated by its effect on the image of a cat in $[0,1] \times [0,1]$.

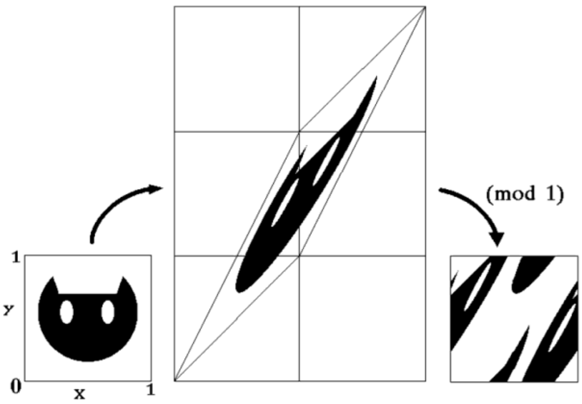
$$\Gamma: \begin{matrix} x_{n+1} = x_n + y_n \bmod 1 \\ y_{n+1} = x_n + 2y_n \bmod 1 \end{matrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} \curvearrowright T \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1 \;\text{ with }\; T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The map shears the image of the unit square $[0,1]^2$ and rearranges the pieces back into the unit square.

Horizontal shear (shear factor h) $T_h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ $\begin{cases} x \to x + h \cdot y \\ y \to y \end{cases}$

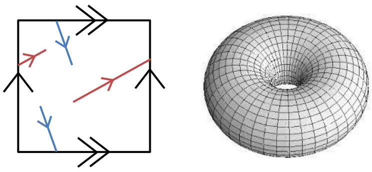
Vertical shear (shear factor v) $T_v = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ $\begin{cases} x \to x \\ y \to y + v \cdot x \end{cases}$



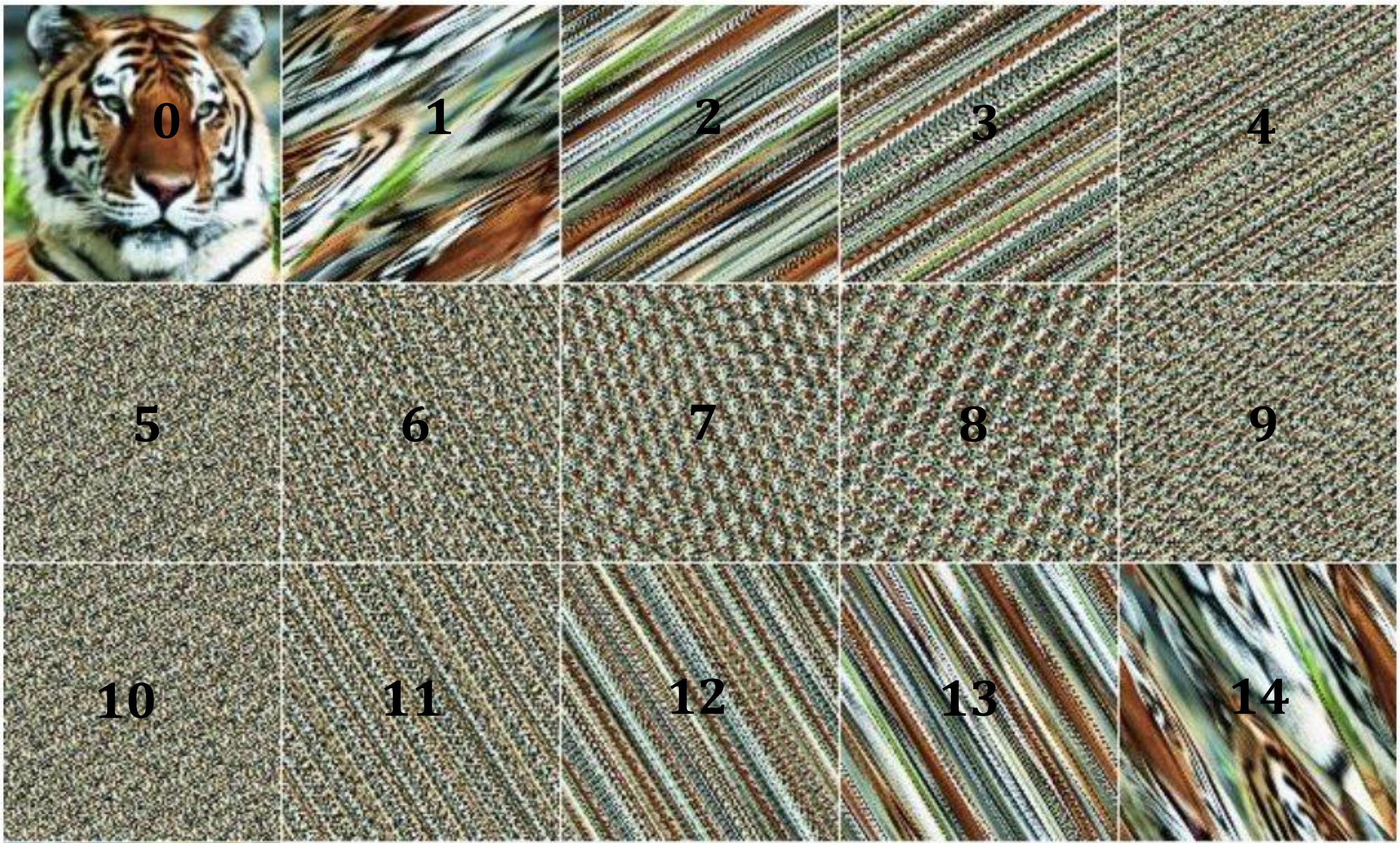


Cat map with horizontal shear followed by vertical shear

The mod operator makes it natural to view the map as acting on a torus with x and y being angular variables where 1 is a full turn. Mathematically a torus is a quotient space, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. $\mathbb{T}^2$ is a square where opposite sides have been identified.



Repeated application of the cat map on a cat image in 124×124 pixels results in:

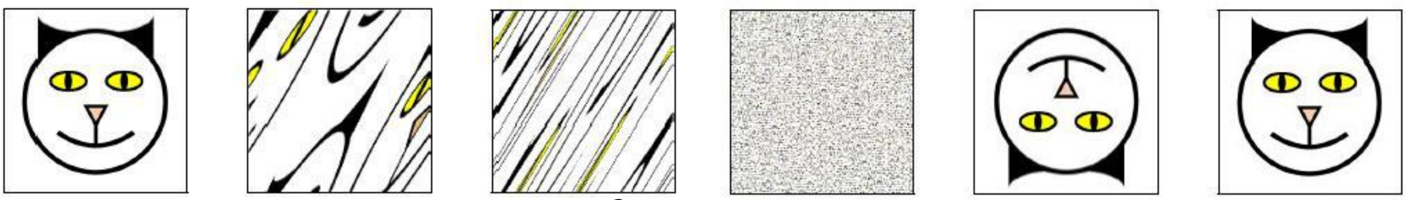


For an image of $N \times N$ pixels where each pixel is represented by rational coordinates $(x/N, y/N)$ and $x, y \in \{0,1,2, \dots N-1\}$ you can use **Arnold's discrete cat map**:

$$\Gamma_N : \mathbb{Z}_N^2 \to \mathbb{Z}_N^2 \quad \text{where} \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \pmod{N} \quad x_n, y_n \in \mathbb{Z}_N$$

Each iteration reshuffles the pixels in the $N \times N$ square.

The conditions of the **Poincaré recurrence theorem** are fulfilled. It states that after a certain number of iterations a dynamical system will come arbitrarily close to (in a continuous system) or will return to exactly the same state (in a discrete system) as the initial state. The number of iterations depends on N.



$n = 0$ $n = 1$ $n = 2$ $n = 6$ $n = 153$ $n = 306$

Arnold's discretized cat map returns after 306 iterations when applied to a 289×289 pixel image.

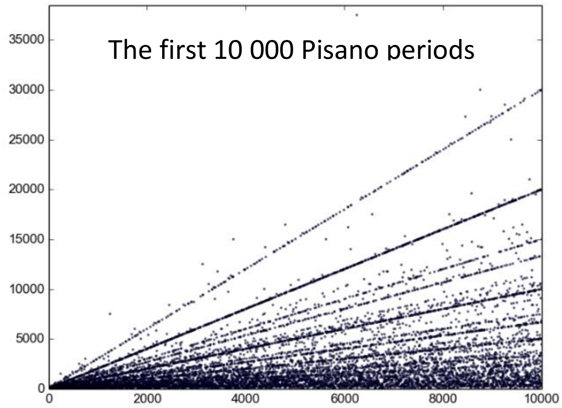
The first reappearance of the original $N \times N$ image under the discretized cat map with $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is the smallest n such that $T^n \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N$, it's the **minimal period of Arnold's discrete cat map modulo N**.
The reappearance number is denoted $\Pi_T(N)$, $\Pi_T(124) = 15$ and $\Pi_T(289) = 306$

The recurrence number is connected to the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765	10946	17711	28657

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix} \rightarrow F^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \qquad F^2 = T \rightarrow T^n = \begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix}$$

$\Pi_T(N)$ is the smallest n such that $\begin{cases} F_{2n-1} \equiv 1 \bmod N \\ F_{2n} \equiv 0 \bmod N \end{cases}$



The Fibonacci sequence modulo N repeats after a certain number of steps called the **Pisano period** $\pi(N)$. The number is named after Leonardo Pisano, Italian mathematician who lived 1170-1250. He is more known known as Fibonacci. $\Pi_T(N)$ is exactly half the Pisano period for all $N > 3$.

Fibonacci sequence mod 124, $\pi(124) = 30$:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	1	2	3	5	8	13	21	34	55	89	20	109	5	114	119	109	104	89	69	34	103	13	116	5	121	2	123	1	0	1

The period can be calculated by prime factorization of $N = p_1^{k_1} p_2^{k_2} \cdot \ldots p_m^{k_m}$.

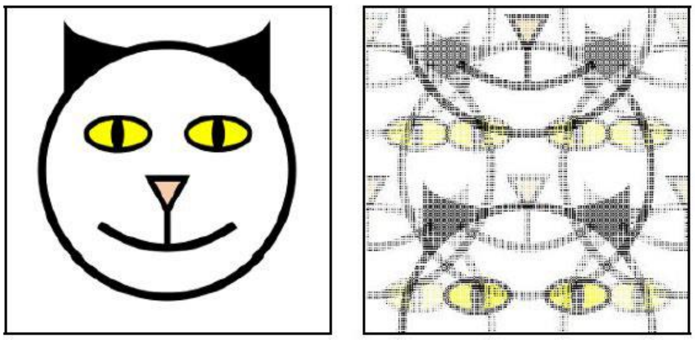
$\Pi_T(N) = \mathrm{LCM}\left(\Pi_T(p_1^{k_1}), \Pi_T(p_2^{k_2}), \ldots, \Pi_T(p_m^{k_m})\right)$

$\Pi_T(124) = \mathrm{LCM}(\Pi_T(4), \Pi_T(31)) = \mathrm{LCM}(3,15) = 15$

$T^3 \bmod 4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

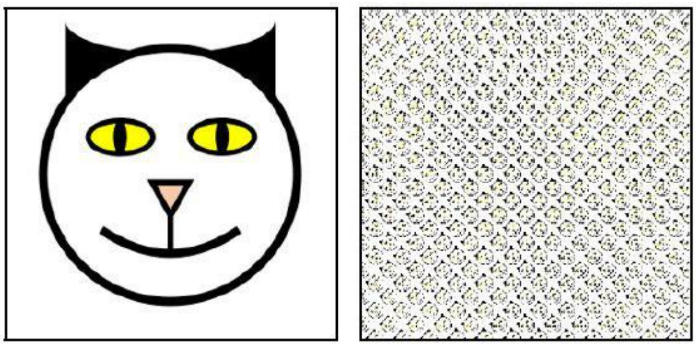
$T^{15} \bmod 31 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Other phenomena connected to the discretized cat map is the appearance of ghosts and miniatures.



Ghosts occurring after 70 iterations on an image of 286^2 pixels

Iteration of image with $\Pi(768) = 192$ and ghosts at $n = 96$



Miniatures after 34 iterations on a cat map under iterations with $T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ on a 289×289 image.

Miniatures occur when the absolute values of elements of $T^n \bmod N$ are small compared to N.

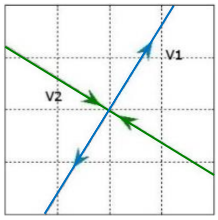
The cat map matrix $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ in $\boldsymbol{r}_{n+1} = T\boldsymbol{r}_n$ mod 1 is symmetric with $\det(T) = 1$ which makes it area-preserving and with orthogonal eigenvectors with product 1.

A more general class of cat maps are given by $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. The maps are categorized as:

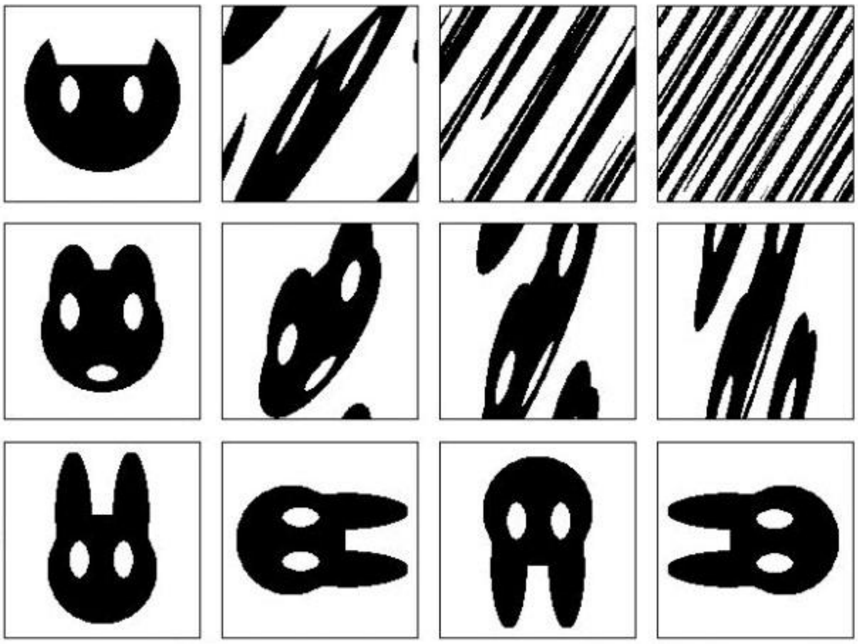
- **Hyperbolic** if one eigenvalue is large than one and the second is less than one.
- **Parabolic** if the eigenvalues satisfy $\lambda_1 = \lambda_2 = 1$
- **Elliptic** if λ_1 and λ_2 are complex conjugate.

The eigenvectors of $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ are $\begin{cases} \boldsymbol{v}_1 = \left(1, \frac{1+\sqrt{5}}{2}\right) \text{ with eigenvalue } \lambda_1 = \frac{3+\sqrt{5}}{2} \approx 2.62 \\ \boldsymbol{v}_2 = \left(1, \frac{1-\sqrt{5}}{2}\right) \text{ with eigenvalue } \lambda_2 = \frac{3-\sqrt{5}}{2} \approx 0.38 \end{cases}$

Arnold's cat map is hyperbolic.



$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$
$\lambda_1 \cdot \lambda_2 = 1$



Hyperbolic case

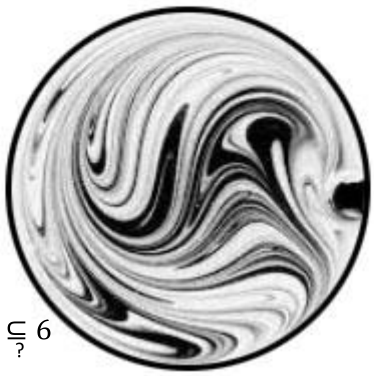
Stretching by factor λ_1 along eigenvector $\boldsymbol{v}_1$
Compression along eigenvector $\boldsymbol{v}_2$ by factor λ_2

Parabolic case

Extension of image preserved in one direction.

Elliptic case

Rotation of image under preserved form.



The cat map (hyperbolic and continuous version) illustrates a property called mixing. A typical example in everyday life is when two different colors of paint are stirred together.

Chaotic systems can be classified after their degree of randomness.

Hierarchy of chaos, in increasing degree of randomness: $1 \subset 2 \subset 3 \subset 4 \subset 5 \underset{?}{\subsetneq} 6$

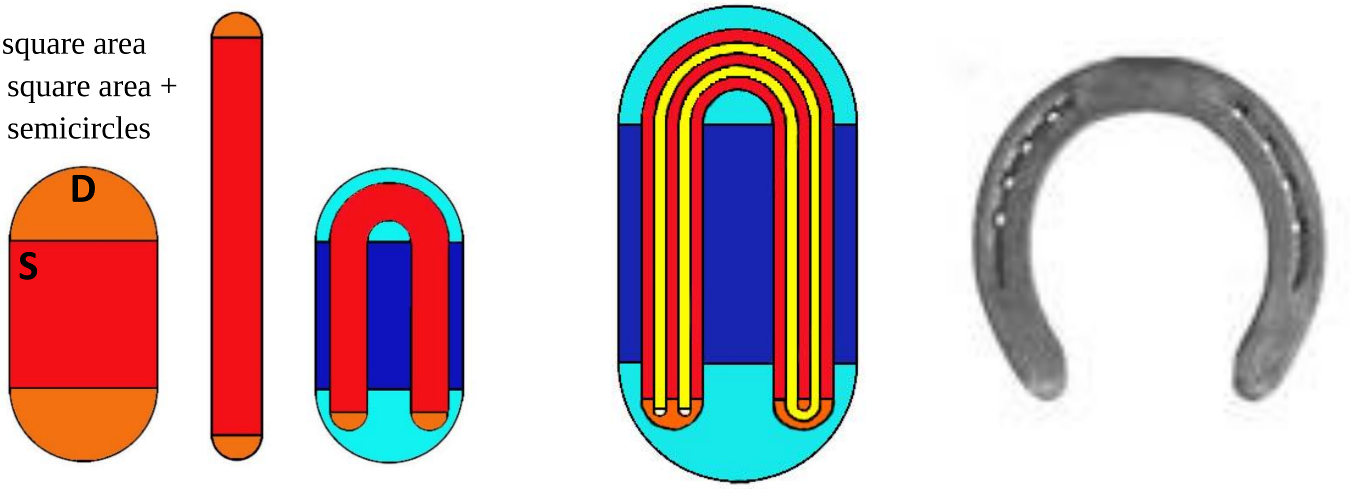
1. Recurrent	The trajectory returns to a given neighborhood of a point an infinite number of times.
2. Ergodic system	Time averages can be replaced by averages over phase space.
3. Mixing	Any area element will eventually spread over the whole phase space
4. K-system	Measure preserving automorphism on probability space obeying Kolomogorov 1-0 law.
5. C-system	All trajectories diverge exponentially in every part of phase space. (ex. Arnold's cat)
6. Bernoulli system	As random as the toss of a coin, can be described by Markov chain. (ex. Random walk)

Horseshoe map

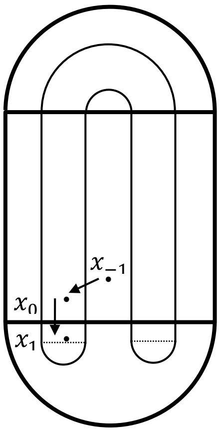
Newton's law of gravitation gives a system of differential equations for the time evolution $\boldsymbol{r}_i(t)$ of a number of objects moving under their mutual gravitational attraction. For two objects the paths are given by easy formulas with simple orbits or trajectories. For three objects the paths are complicated and unpredictable unless you iterate the equations forward one step at a time. There are no closed for formulas for their solution. The French mathematician **Henri Poincaré** worked on the **3-body problem** and tried to characterize the motion. He found chaos in the solution of nonlinear differential equations and won a big prize for it in 1887.

It took a long time to get order in the chaos that Poincaré discovered. A big step was taken by Stephen Smale in the 1960's when he derived a geometric view to understand the complicated patterns and motions in nonlinear dynamical systems. His invention is known as **Smale's horseshoe map**.

S: square area
D: square area + semicircles



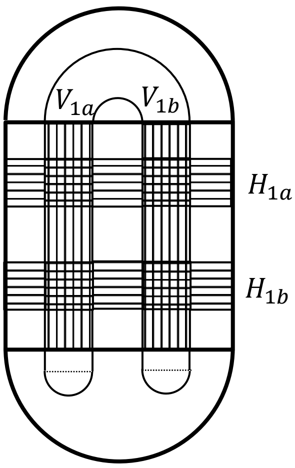
Take a square with semicircles on top and bottom, squeeze the square →←, stretch it ↕ and fold it ∩ back on top of the original area. Repeat the procedure of squeeze, stretch and fold repeatedly.



A point $x_0 \in D$ is moved to $x_1 = f(x_0)$ and then to $f_2 = f^2(x_0)$ by the horseshoe map f in an infinite sequence, the **forward orbit** of x_0: $x_0, x_1 x_2, \ldots$.

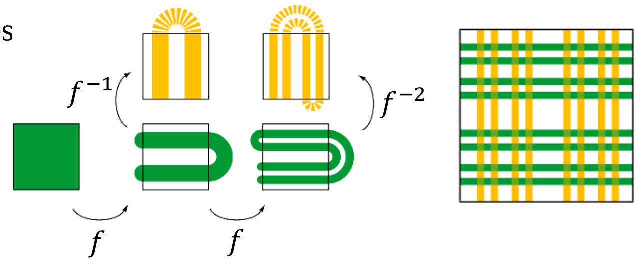
There is also a backward orbit but not all points in D have an inverse image x_{-1} which might have an inverse image x_{-2} leading to a finite or infinite **backward orbit** of x_0: $x_{-1}, x_{-2}, x_{-3}, \ldots$

Just as some points in S don't have an inverse there are some points in S that are mapped to regions outside S. Points that are mapped into S lie in two horizontal strips H_0 and H_1. These strips are mapped into two vertical strips V_0 and V_1, $f(H_{1a} \cup H_{1b}) = V_{1a} \cup V_{1b}$.



Which points have infinite forward and infinite backward orbits contained in S? They must lie in the intersection $(H_{1a} \cup H_{1b}) \cap (V_{1a} \cap V_{1b})$, the 4 chequered squares.

Follwing the iteration gives a gives a series of diminishing horisontal $H_n = f^n(S) \cap S$ and vertical $V_n = f^{-n}(H_n)$ stripes The intersection of H_n and V_n converges to an invariant set $\Lambda = \lim_{n\to\infty} H_n \cap V_n$ of points with Cantor-like distribution in both vertical and horizontal directions.



There is a **symbolic dynamic** of how points in the set Λ are mapped into each other. The dynamics help to understand the dynamics of complex nonlinear dynamical systems.

$x_0 \in \Lambda \Rightarrow x_n, x_{n+1} \in \Lambda$ and x_{n+1} must lie in either V_{1a} or V_{1b} which means that x_n must lie in H_{1a} or H_{1b}. This is true for every n, both in the forward and backward orbit. Write down a 0 of x_n is in H_{1a} and 1 if x_n is in H_{1b}. This gives a bi-infinite sequence of 0s and 1s ... $a_{-2}a_{-1}.a_0a_1a_2$ ... which corresponds one-to-one with points in Λ. Every point in Λ has a unique bi-infinite sequence and vice versa.

The horse shoe map f on $x \in \Lambda$ corresponds to shifting the dot in the bi-infinite sequence one step. Numbers in Λ that are close to each other will agree in their sequences for a large number of places to the right and left of the dot. The closer they are the more places agree.

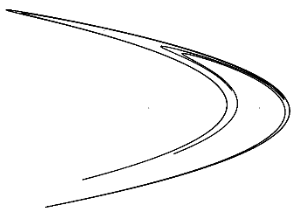
Attractors

An attractor is a set of states toward which a dynamical system evolve, a form of limit set for a system that can be discrete $\boldsymbol{r}_{t+1} = f(\boldsymbol{r}_t)$ or continuous $\dot{\boldsymbol{r}} = f(\boldsymbol{r})$. The attractor is stable in the sense that when $\boldsymbol{r}$ gets close enough to the attractor it remains close.

The attractor can be a limiting point or cycle, a curve or a manifold but it can also be a complicated geometrical structure in n-dimensional space with non-integer Hausdorff dimension, a **strange attractor**.

In the discrete case the system can be described by a difference equation. In the continuous case the description is based on differential equations. These are handled in the next section.

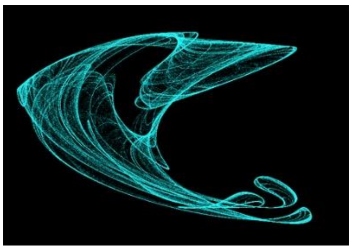
Discrete



Hénon attractor

$$\begin{cases} x_{t+1} = 1 - ax_t^2 + y_t \\ y_{t+1} = bx_t \end{cases}$$

$a = 1.4 \;\; b = 0.3$



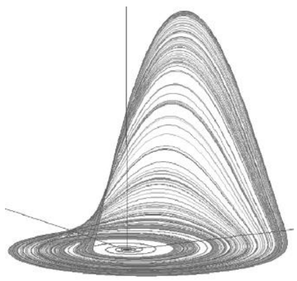
De Jong attractor

$$\begin{cases} x_{t+1} = \sin(ay_t) - \cos(bx_t) \\ y_{t+1} = \sin(cx_t) - \cos(dy_t) \end{cases}$$

$a = 2.01 \;\; b = -2.53$

$c = 1.61 \;\; d = -0.33$

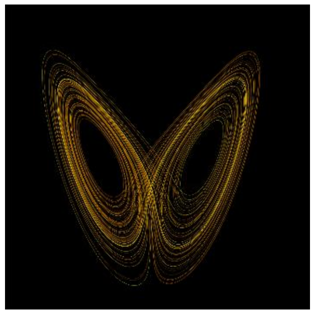
Continuous



Rössler attractor

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$$

$a = 0.1 \;\; b = 0.1 \;\; c = 14.0$



Lorentz attractor

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$

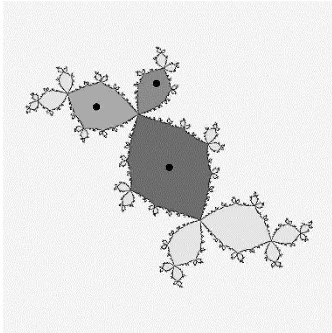
$\rho = 28 \;\; \sigma = 10 \;\; \beta = 8/3$

Attractors (A) are subsets of the phase space to the dynamical system. They have a **basin of attraction** $B(A)$ in which every orbit approaches the attractor and which contain an open set containing the attractor.

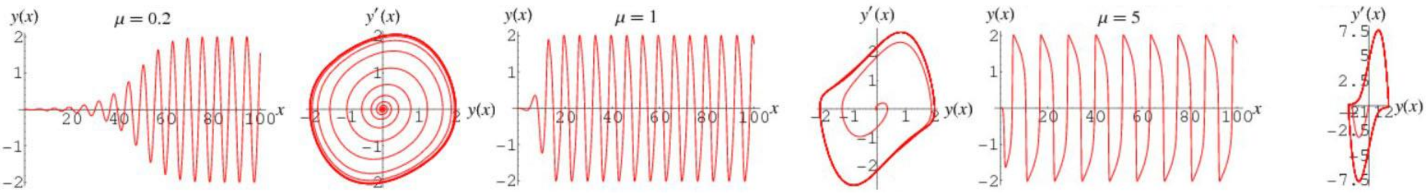
Strange attractors like the Lorentz attractor where first thought to be exceptional and fragile. Stephen Smale showed with his horseshoe map that strange attractors can be robust and that they have a Cantor-like structure, which means that there are intersections of the attractor with suitable planes that are totally disconnected and noncountable.

Attractors can be **fixed points** that maps to themselves like the end position of a damped pendulum. The pendulum in the unstable upward position is an unstable equilibrium and therefore not an attractor, it's a **repeller**.

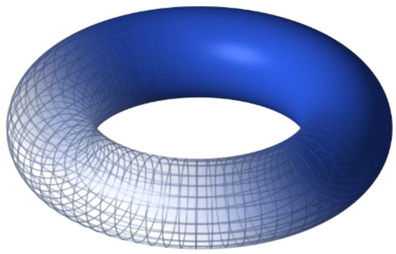
Discrete systems can have **limit cycles** that are periodic orbits $\boldsymbol{r}_1 \to \boldsymbol{r}_2 \to \cdots \to \boldsymbol{r}_n$ like the 3-cycle in a Julia set to $z \to z^2 + c$ where each iteration with $z_0 \in B(A)$ leads to an asymptotic convergence to the 3-cycle. (Exception exist, non-hyperbolic). Picture shows basin of attraction to limit cycle consisting of three points. Staring in bulbs outside the 3 grey bulbs jump around in other bulbs before settling down in the inner trio and then stat convergence to the limit cycle.



In the continuous case limit cycles can be in the form of closed loops. A clock pendulum with an energy feed that maintains a stable oscillation is an example. This is governed by a differential equation with an oscillating driving term $\ddot{x} + A(x)\dot{x} + B(x) = f(x,t)$. Other examples can be without driving terms like the Van der Pool oscillator with non-linear damping $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$. A non-conservative system, the work done in moving a particle between two points depends on the path taken. The differential equation was produced to describes vacuum tubes. It's an equation that describes self-sustaining oscillation where energy is removed from large oscillations and fed into small oscillations.

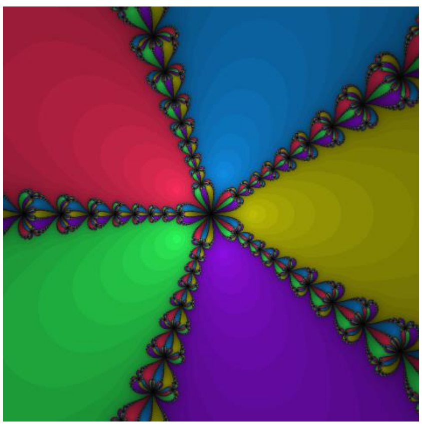


In the caser of multiple frequencies in the limiting cycle and where the ratios of the frequencies are irrational the limit orbit takes place on a torus and the orbit will not be closed.
The limit cycle will fill the torus and the attractor will be an n-torus if there are n frequencies that are incommensurate.

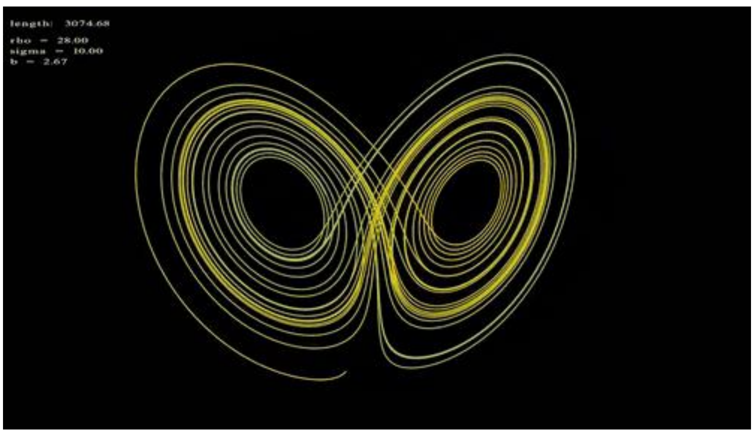


Apart from the fixed point, limit cycle and limit torus there are the strange attractors with fractal geometry. Theses attractors are usually connected to chaotic dynamics with sensitive dependence on initial conditions and exponential separation. The system has stretch and fold properties that contain the attractor.

A dynamic system with a strange attractor is locally unstable and globally unstable. Points on the attractor will diverge from each other but never depart from the attractor.



5 basins of attraction in $\mathbb{C}$, in separate colors to find 5 roots to $z^5 = 1$ with Newton's method starting at $z_0 \in \mathbb{C}$ and iterating towards some solution.



Strange attractor of the Lorentz differential equation. Solutions $\boldsymbol{r}(t)$ with some starting point $\boldsymbol{r}(t_0)$ outside the attractor will converge to the Lorentz attractor, a set in $\mathbb{R}^3$ with fractal Hausdorff dimension 2.06 ± 0.01.